Konzept selbstorganisierende Oszillation
Sections 7.3 and 7.4 in [Mur02]

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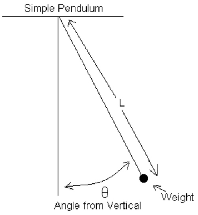
- Ja!

Inhalt

Pendel und DGLs: Zielsetzung

- Intuition
- mathematische Einführung mit minimalen Vorkenntnissen

Pendel



$$\text{Änderung des Winkels } \frac{d\theta}{dt} = \omega \tag{1}$$

$$\text{Änderung der Winkelgeschwindigkeit} \frac{d\omega}{dt} = -c_1 \sin\theta + \underbrace{f_{\text{drag}}(\omega)}_{\approx -c_2\omega \text{ für } \omega \text{ klein}} \tag{2}$$

$(c_1, c_2 > 0$ sind Konstanten$)$

Ziel

- Equilibrium finden:

$$0 \overset{!}{=} \frac{d\theta}{dt} = \omega \tag{3}$$

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- Wie kann man diesen Unterschied mathematisch sehen?

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$$\frac{d\bar{\omega}}{dt} = -c_1 \underbrace{\sin\bar{\theta}}_{\text{Taylor!}} - c_2\bar{\omega} \tag{6}$$

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$$\text{Taylor around } x = 0\text{: } \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{7}$$

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$$\lambda_{1/2} = -\frac{1}{2}c_2 \pm i\sqrt{4c_1 - c_2^2} \quad (8)$$

- Recall: Eigenvalues: $\lambda_{1/2} = -\frac{1}{2}c_2 \pm i\sqrt{4c_1 - c_2^2}$.
- Solution: $\begin{pmatrix} \bar{\theta} \\ \bar{\omega} \end{pmatrix} = d_1\vec{X}_1 e^{\lambda_1 t} + d_2\vec{X}_2 e^{\lambda_2 t}$ where d_1, d_2 are determined by initial conditions.
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- Repeating steps for $(\pi, 0)$ we won't get convergence (unstable)

Linear Stability Analysis: General Framework

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$$\frac{d}{dt}\begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}}_{=:A} \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} \tag{11}$$

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To be continued...

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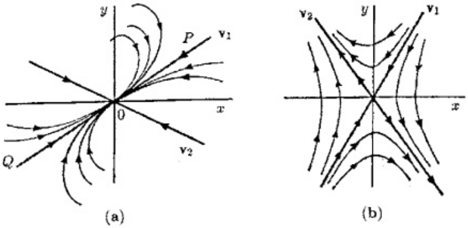
$$c_1 \vec{X}_1 e^{\lambda_1 t} + c_2 \vec{X}_2 e^{\lambda_2 t} \tag{12}$$

$$= e^{\mathrm{Re}(\lambda_1)t} e^{i\mathrm{Im}(\lambda_1)t} + e^{\mathrm{Re}(\lambda_2)t} e^{i\mathrm{Im}(\lambda_2)t} \tag{13}$$

λ_1, λ_2 determine type of steady state:

- If λ_1, λ_2 both real and distinct:
  1. $\mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2) = \begin{cases} - & \text{stable node, see Figure (a)} \\ + & \text{unstable node} \end{cases}$
  2. $\mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$: saddle point (unstable), see Figure (b)



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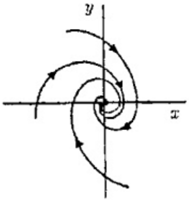
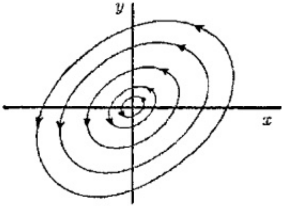
λ_1, λ_2 determine type of steady state:

- If λ_1, λ_2 complex: $\lambda_{1/2} = \alpha \pm i\beta$ $(\beta \neq 0)$
  1. If $\alpha \neq 0$: $\mathrm{sign}(\alpha) = \begin{cases} - & \text{stable} \\ + & \text{unstable} \end{cases}$ spiral, see Figure (c)
  2. $\alpha = 0$: ellipses, see Figure (d)



(c)



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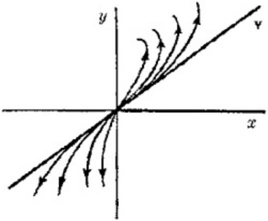
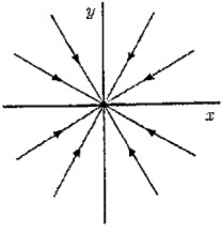
λ_1, λ_2 determine type of steady state:

- $\lambda_1 = \lambda_2$. Can be stable or unstable.

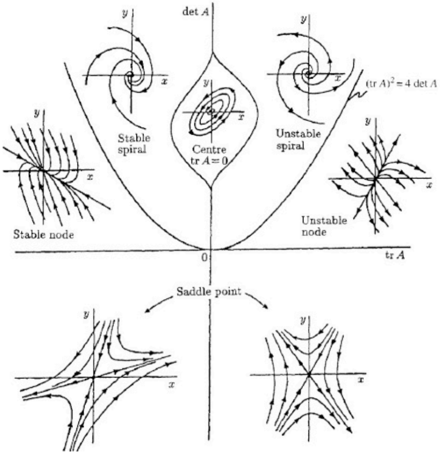


(e)



(f)

Summary: Types of equilibria



Recall: $A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{pmatrix}\Bigg|_{(u_0,v_0)}$

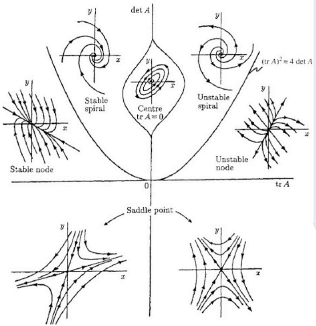
Stable iff

$tr(A) = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v} < 0$

and

$det(A) = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u} > 0$

Oscillation



Confined set picture.

Theorem 1 (Poincaré-Bendixson)

If system possesses a confined set enclosing a single singular point which is an unstable spiral or node then any phase trajectory cannot tend to the singularity with time, nor can it leave the confined set. As $t \to \infty$, the trajectory will tend to a limit cycle solution (Oscillation).

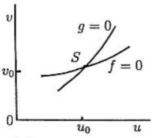
For an unstable node or spiral to occur:

$$\operatorname{tr} A > 0, \det A > 0, \quad (14)$$

$$\operatorname{tr} A^2 \begin{cases} > 4|A| \\ < 4|A| \end{cases} \Rightarrow \text{unstable} \begin{cases} \text{node} \\ \text{spiral.} \end{cases}$$

Conditions on matrix A

Want conditions on A to have limit cycle solution (oscillation).
Recall: $\mathrm{tr}\, A > 0, \det A > 0$ is sufficient.



(a)

- Null clines: Intersection at equilibrium
- Suppose have confined set around S

- Gradient $\partial v/\partial u > 0$ with (Gymnasium)

$$dv/du]_{g=0} > dv/du]_{f=0} > 0 \qquad (16)$$

- Using chain rule (details), get identities:

$$dv/du]_{g=0} = -\frac{\partial g/\partial u}{\partial g/\partial v} \qquad (17)$$

$$dv/du]_{f=0} = -\frac{\partial f/\partial u}{\partial f/\partial v} \qquad (18)$$

$$\Rightarrow \; -\frac{\partial g/\partial u}{\partial g/\partial v} > -\frac{\partial f/\partial u}{\partial f/\partial v} > 0$$

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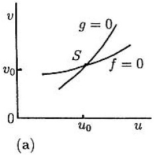

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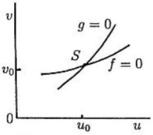
- Recall: $-\frac{\partial g/\partial u}{\partial g/\partial v} > -\frac{\partial f/\partial u}{\partial f/\partial v} > 0$.
- $\det A = \partial f/\partial u \cdot \partial g/\partial v - \partial f/\partial v \cdot \partial g/\partial u > 0$ if $\operatorname{sign}(\partial g/\partial v) = \operatorname{sign}(\partial f/\partial v)$ (multiplication).



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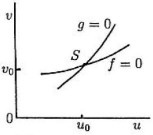

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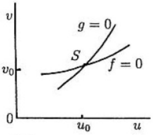

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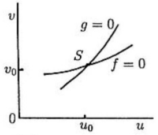

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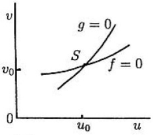

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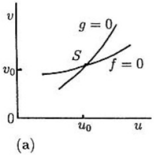

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- Further analysis would require to know the signs of $\partial f/\partial u, \dots$

Parameter Domain Determination for Oscillations

- Goal: Get conditions on parameters to have periodic solutions

[1] using the Law of Mass Action

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$$X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} A, \quad B \xrightarrow{k_2} Y, \quad 2X + Y \xrightarrow{k_3} 3X$$

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where a, b are positive constants.

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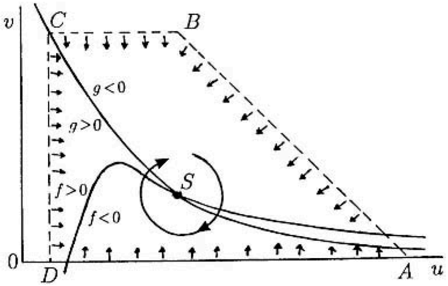
  where a, b are positive constants.
- For which a, b do we have oscillatory solutions?

[1] using the Law of Mass Action

1. Mathematical system of (non-linear) ODEs
$\frac{du}{dt} = a - u + u^2 v = f(u,v), \; \frac{dv}{dt} = b - u^2 v = g(u,v)$



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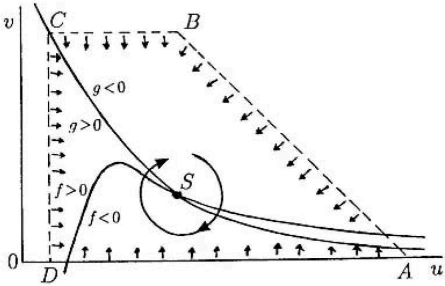

   Assume confined set around S.
2. Find equilibria

$$f(u_0, v_0) = a - u_0 + u_0^2 v_0 \stackrel{!}{=} 0, g(u_0, v_0) = b - u_0^2 v_0 \stackrel{!}{=} 0$$

$$\Rightarrow u_0 = b + a, v_0 = \frac{b}{(a+b)^2}, \text{with } b > a, a + b > 0.$$

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   - $\det A = u_0^2 > 0$ ✓
   - $\operatorname{tr} A = \partial f/\partial u + \partial g/\partial v > 0$: $\Rightarrow 1 - 2a/u_0 - u_0^2 > 0$
     $\Rightarrow a < \frac{u_0(1-u_0^2)}{2},\ b = u_0 - a > \frac{u_0(1+u_0^2)}{2}$. ✓

Conditions on a, b to have limit cycle solution:

$$a < \frac{u_0(1-u_0^2)}{2} \tag{20}$$

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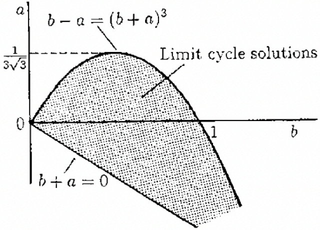

Summary

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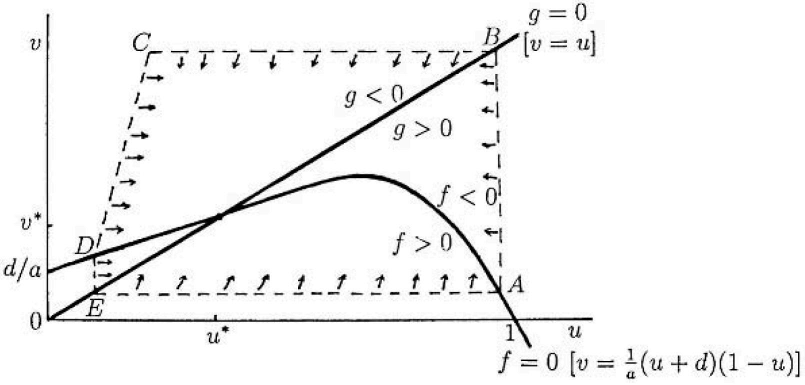

Back to equilibria overview.